Planetoid Strings : Solutions and Perturbations

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Abstract

A novel ansatz for solving the string equations of motion and constraints in generic curved backgrounds, namely the *planetoid ansatz*, was proposed recently by some authors. We construct several specific examples of planetoid strings in curved backgrounds which include Lorentzian wormholes, spherical Rindler spacetime and the 2+1 dimensional black hole. A semiclassical quantisation is performed and the Regge relations for the planetoids are obtained. The general equations for the study of small perturbations about these solutions are written down using the standard, manifestly covariant formalism. Applications to special cases such as those of planetoid strings in Minkowski and spherical Rindler spacetimes are also presented.

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I. INTRODUCTION

In the context of cosmic as well as fundamental strings, the analysis of the classical string equations of motion and constraints in generic curved backgrounds [1] has become an active area of research over the last decade or so (for a recent review and references see [2]) Solutions representing string configurations, which essentially correspond to timelike embedded minimal surfaces, are difficult to obtain largely due to the nonlinear and coupled nature of the relevant equations. Therefore, the attitude has been to proceed by proposing a generic ansatz based on symmetries or simplifying assumptions which reduce the complicated set of equations to a tractable form. Among the various proposals till date, we have the stationary string ansatz [3], dynamic circular strings [4] and more recently, planetoid string configurations [5] as well as rigidly rotating strings [6]. It is worth mentioning that the planetoid and rigidly rotating strings are both special cases of an ansatz proposed earlier by Larsen and Sanchez [7]. Target spaces with metrics such as the Schwarzschild, Kerr-Newman, Robertson–Walker, cosmic strings, wormholes etc. have been chosen and explicit string configurations obtained in these backgrounds. Once specific configurations are known, the obvious next question that emerges is about their stability. This turns out to be related to the second variation of the action (Nambu–Goto or its generalisations) and the corresponding Jacobi equations [9]. Perturbative stability depends crucially on the analysis of these equations. String propagation in an exact, stringy four-dimensional black hole background and the perturbations about extremal configurations have also been studied recently [8]. Furthermore, nonperturbative effects which include the formation of cusps and kinks on the world surface of the string is governed by the character of solutions of the generalised Raychaudhuri equations [10].

This paper deals with planetoid strings. First we obtain specific solutions in certain well known backgrounds. Thereafter, we discuss small perturbations about these configurations. The backgrounds chosen include the Ellis geometry (a Lorentzian wormhole), the spherical Rindler spacetime, the Minkowski spacetime and the 2+1 dimensional BTZ black hole [11].

We also consider semi-classical quantization of strings in these backgrounds and compute physical quantities such as the classical action, mass, reduced action and angular momentum. The quantisation condition for each case is written explicitly.

Notations and sign conventions in the paper follow the norms of Misner, Thorne and Wheeler [12].

II. PLANETOID STRINGS: FORMALISM

We first briefly discuss general planetoid strings, quote the ansatz and the resulting equations which we solve for specific backgrounds later.

The generic background metric (taking a $\theta = \frac{\pi}{2}$ section) is taken to be of the form :

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 \tag{1}$$

The planetoid ansatz is given as [5],

$$t = t_0 + \alpha\tau \quad ; \quad \phi = \phi_0 + \beta\tau \quad ; \quad r = r(\sigma) \tag{2}$$

where, τ and σ are the time-like and space-like coordinates on the worldsheet respectively. α and β are two arbitrary constants. Assuming $\beta = 0$ would give us the usual stationary strings. Note that the planetoid ansatz is a special case of the one proposed by Larsen and Sanchez [7] where the constants t_0 and ϕ_0 are replaced by general functions $t_0(\sigma)$ and $\phi_0(\sigma)$ respectively. A word about the name 'planetoid'. The ansatz above is a sort of generalisation of the ansatz one would take if one deals with the embedded curves along which planets move in their orbit. Hence it is perhaps appropriate to call these kinds of worldsheets 'planetoids' – a name which drives home the message that these are related to planetary orbits while being surfaces as opposed to curves.

From the bosonic string equations of motion and constraints one arrives at the first order equation, which one needs to solve in order to get a planetoid string. This is given as :

$$\left(\frac{dr}{d\sigma}\right)^2 = -g^{rr}\left[\alpha^2 g_{tt} + 2\alpha\beta g_{t\phi} + \beta^2 g_{\phi\phi}\right] \tag{3}$$

where the right hand side can be identified with the negative of a potential $V(r)$. However, it is more convenient to work with $\tilde{V}(r)$ which is defined as,

$$\tilde{V}(r) = \frac{V(r)}{\alpha^2} = g^{rr}\left[g_{tt} + \frac{2\beta}{\alpha}g_{t\phi} + \frac{\beta^2}{\alpha^2}g_{\phi\phi}\right] \tag{4}$$

The induced metric on the world-sheet of the string is given as :

$$ds_I^2 = \left(\frac{dr}{d\sigma}\right)^2 g_{rr}\left[-d\tau^2 + d\sigma^2\right] \tag{5}$$

By choosing a conformal gauge in which the induced metric is diagonal and conformal to Minkowski spacetime in two dimensions, we automatically satisfy the constraint equations $(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + g_{\mu\nu}x'^\mu x'^\nu = 0; \quad g_{\mu\nu}\dot{x}^\mu x'^\nu = 0)$, where dot and prime denote differentiations with respect to world-sheet coordinates τ and σ respectively and μ, ν are space-time indices. We confine ourselves largely to spherically symmetric, static backgrounds for which the basic equation to solve turns out to be :

$$\frac{dr}{d\sigma} = \pm\sqrt{\left(1 - \frac{b(r)}{r}\right)\left[\alpha^2 e^{2\psi(r)} - r^2\beta^2\right]} \tag{6}$$

where our background metric is now assumed as diagonal and for a $\theta = \frac{\pi}{2}$ section, it is given as :

$$ds^2 = -e^{2\psi(r)}dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\phi^2 \tag{7}$$

When is the induced metric on a planetoid string Minkowskian? By looking at the expression for the induced metric one can easily say that this happens if :

$$\left(\frac{dr}{d\sigma}\right)^2 = C^2 g^{rr} \tag{8}$$

For spherically symmetric, static metrics this turns out to be a very stringent constraint on the red–shift function $\psi(r)$, which should satisfy,

$$e^{2\psi(r)} = \frac{C^2 + r^2\beta^2}{\alpha^2} \tag{9}$$

Additionally, we observe that the existence of a zero in the conformal factor in the induced metric would indicate the existence of a singularity on the worldsheet. Specifically, if $r = r_0$ is a zero of the expression for the conformal factor we must have :

$$e^{2\psi(r_0)}\alpha^2 = \beta^2 r_0^2 \tag{10}$$

If r_0 coincides with the horizon $e^{2\psi(r_0)} = 0$ then we can only have $r_0 = 0$. There maybe other points in the geometry where this could be satisfied too regardless of whether the geometry has a horizon or not. On the other hand if $e^{2\psi} = 1$ (i.e. an ultrastatic metric) we can clearly see that $r = \frac{\alpha}{\beta}$ is the point where the worldsheet will become singular. These facts will be generic features of all the solutions to be discussed below.

Let us also consider planetoids in generic time–dependent backgrounds of the form :

$$ds^2 = \Omega^2(t)\left(-dt^2 + \frac{dr^2}{1-\frac{b(r)}{r}} + r^2 d\Omega_2^2\right) \tag{11}$$

It can be shown that there will be no planetoid solutions in time-dependent backgrounds of the above type (which includes the FRW models too). To see this let us look at the string equation of motion for the coordinate ϕ. With the substitution of the planetoid ansatz, we find that the equation reduces to the requirement :

$$-\alpha\beta\frac{\dot{\Omega}}{\Omega} = 0 \tag{12}$$

Since α or β cannot be taken to be zero one needs $\Omega(t)$ to be a constant.

Also note that the planetoid ansatz is incompatible with null (tensionless) strings as has been pointed out in [13].

We now move on towards solving the string equation of motion to obtain specific planetoid string configuration in some well–known backgrounds.

III. SOLUTIONS IN SPECIFIC BACKGROUNDS

(1) Spherically symmetric coordinate representation of Minkowski spacetime

In this case, the background metric (as given in the form in eqn. (7)) has $b(r) = 0$ and $\psi(r) = 0$. The planetoid solution is :

$$r = \frac{\alpha}{\beta}|\sin\beta\,(\sigma-\sigma_0)\,| \tag{13}$$

The expression for $r(\sigma)$ is the modulus of the sine function. One needs to consider the absolute value in order to define the worldsheet with proper Neumann–type boundary conditions at the edge values $\sigma = \sigma_0 \pm \frac{\pi}{2\beta}$. This, ofcourse results in a *kink* in the metric and consequently a δ–function curvature singularity at $\sigma = \sigma_0$.

The induced metric at all points $\sigma \neq 0$ on the world-sheet is given as :

$$ds^2 = \alpha^2 \cos^2 \beta(\sigma - \sigma_0) \left(-d\tau^2 + d\sigma^2\right) \tag{14}$$

which is certainly not flat.

Also note that the metric on the worldsheet has a singularity at $\sigma = \sigma_0 \pm \frac{\pi}{2\beta}$. This can be checked easily by calculating the Ricci scalar which turns out to be :

$${}^2R = \frac{2\beta^2}{\cos^4 \beta\,(\sigma - \sigma_0)} \tag{15}$$

where we have excluded the δ–function contribution at $\sigma = 0$.

Therefore, it is necessary to restrict the worldsheet to the domain $-\frac{\pi}{2\beta} < \sigma - \sigma_0 < \frac{\pi}{2\beta}$.

(This planetoid solution is obtained in [5], but the important point is to realise is that the induced metric is not flat, is singular and therefore, as we shall see in the later sections of this paper the perturbation equations will be nontrivial by virtue of the $K^i_{ab}K^{ab}_i$ term even though the Riemann tensor term makes no contribution.)

(2) Spherical Rindler Spacetime

Here $e^{2\psi(r)} = r^2$ and $b(r) = 0$. A word about the spherical Rindler spacetime. This is the generalisation of the usual Rindler spacetime written in Cartesian coordinates to a spherisymmetric metric. The matter stress energy which would be required to support such a geometry is however somewhat strange. Firstly, spherical Rindler spacetime is not a flat geometry–it has curvature. This can be checked by calculating the Riemann tensor components. Moreover, the nonzero components of $G_{\mu\nu}$ (in the frame–basis) are given as :

$$G_{00} = 0 \qquad ; \qquad G_{11} = \frac{2}{r^2} \qquad ; \qquad G_{22} = G_{33} = \frac{1}{r^2} \tag{16}$$

Defining the energy–momentum tensor $T_{\mu\nu}$ as given by the $G_{\mu\nu}$ via Einstein's field equations, one can see that the matter required to support spherical Rindler spacetime has an

equation of state $\tau = 2p$ and also obeys the Weak Energy Condition (i.e. if ρ, τ, p, p are the diagonal components of the $T_{\mu\nu}$ then one must have $\rho \geq 0, \rho + \tau \geq 0, rho + p \geq 0$).

The horizon of the spacetime ($r = 0$) is also a singular point in the sense of diverging Riemann curvature.

The planetoid solution in this spacetime looks as follows:

$$r(\sigma) = \exp\left(\pm\sqrt{(\alpha^2 - \beta^2)}(\sigma + \sigma_0)\right) \tag{17}$$

with $\beta^2 < \alpha^2$. The solution with the $+$ sign is valid for $\sigma \leq 0$ while the one with the $-$ sign is for the domain $\sigma > 0$. These two solutions ofcourse match at $\sigma = 0$ smoothly. The derivatives of $r(\sigma)$ however are not continuous at $\sigma = 0$. However, note that the induced metric contains $\left(\frac{dr}{d\sigma}\right)^2$ and therefore the metric functions as well as the extrinsic curvatures match smoothly across $\sigma = 0$. We need to have two different solutions in the two domains of σ in order to satisfy the open string Neumann–type boundary conditions at $\sigma = \pm\infty$. The string worldsheet here stretches from $+\infty$ to $-\infty$ and describes a *folded* string with the fold at $\sigma = 0$. One can also adopt the viewpoint that the above solution (say, the $+$ one) describes a semi–infinite string stretching from $r = 0$ to infinity with its only boundary at $r = 0$ ($\sigma \to -\infty$). With $\beta^2 > \alpha^2$ we have oscillatory solutions (they must be complex and therefore are devoid of any physical relevance) and for $\beta^2 = \alpha^2$, $r(\sigma)$ is linear. The worldsheet metric, however, is now flat. This is easily seen by looking at the conformal factor in the induced metric which goes as $e^{\pm\sqrt{\alpha^2-\beta^2}\sigma}$. For a general 2D metric with a conformal factor $e^{2\rho}$ we have ${}^2R = -2e^{-2\rho}\partial_+\partial_-\rho$. Since ρ is linear in this case we have ${}^2R = 0$ straightaway. However, recall the fact that spherical Rindler spacetime in contrast to Rindler spacetime in Cartesian coordinates is a curved space-time and nonzero components of the Riemann tensor do exist. Thus, spacetime curvature will contribute to the perturbations, as we shall see later on.

(3) Ellis geometry

For this case, we have $\psi(r) = 0$ and $b(r) = \frac{b_0^2}{r}$ in the spherically symmetric metric. This geometry represents a traversable Lorentzian wormhole and was discussed first by Ellis [14]

(see also Morris and Thorne [15]). The throat of the wormhole (which corresponds also to the minimum value of r) is at $r = b_0$. As is easily seen, the spacetime is asymptotically flat–two asymptotic regions are connected by the wormhole tunnel. -

After explicit integration of the equation for $r(\sigma)$ we get the following expression for the planetoid configuration :

$$r^2(\sigma) = \frac{1}{2}(A-B)\{\frac{A+B}{A-B} \pm \sin 2\beta(\sigma-\sigma_0)\} \tag{18}$$

where $A = b_0^2, B = \frac{\alpha^2}{\beta^2}$.

If $A > B$ then $\frac{A+B}{A-B} > 1$ and both the $+$ and $-$ solutions are valid. However, in this case $\frac{\alpha^2}{\beta^2} < r^2 < b_0^2$. But one needs $r \geq b_0$, otherwise the spacetime loses its Lorentzian signature. Therefore, this solution is not possible physically. On the other hand if $A < B$ then we have $b_0^2 < r^2 < \frac{\alpha^2}{\beta^2}$ and the planetoid string extends from the minimum value of r (i.e. b_0) upto the value $\frac{\alpha}{\beta}$ which is perfectly allowed. The open string boundary conditions are satisfied at the values $\sigma = \pm\frac{\pi}{4\beta}$ which correspond to r values b_0 and $\frac{\alpha}{\beta}$.

The induced metric on the worldsheet is once again not flat and is given (for the case $\frac{\alpha^2}{\beta^2} > b_0^2$) by :

$$ds_I^2 = \frac{1}{2}\beta^2\left(\frac{\alpha^2}{\beta^2} - b_0^2\right)(1 \mp \sin 2\beta\sigma)\left[-d\tau^2 + d\sigma^2\right] \tag{19}$$

The conformal factor becomes zero at the points $\sigma = \pm\frac{\pi}{4\beta}$ for the $-$ and $+$ solutions respectively. This corresponds to a worldsheet singularity at those points. In spacetime, the worldsheet singularity coincides with the location of the throat at $r = b_0$.

(4) 2+1 dimensional black hole

Here, we consider the $2+1$ dimensional black hole anti-de Sitter space-time obtained by Banados, Teitelboim and Zanelli [11]. The metric is given by,

$$ds^2 = \left(M - \frac{r^2}{l^2}\right)dt^2 + \frac{1}{\left(\frac{r^2}{l^2} - M + \frac{J^2}{4r^2}\right)}dr^2 + r^2 d\phi^2 - J dt d\phi \tag{20}$$

where, M is the mass and J is the angular momentum of the black hole. For simplicity, we take the $M = 1$ and $J = 0$ limit of this metric. In this limit, there is only one horizon at $r = l$. The equation of motion obtained as a first order equation is given as,

$$\left(\frac{dr}{d\sigma}\right)^2 = \left(\frac{r^2}{l^2} - 1\right)\left[\alpha^2\left(\frac{r^2}{l^2} - 1\right) - \beta^2 r^2\right] \tag{21}$$

which is solved in terms of incomplete elliptic integrals [16]. The solution is given by,

$$\pm\left(\sigma - \sigma_0\right) = \frac{l}{\alpha} F(arcsin\left(\frac{r}{l}\right), k) \tag{22}$$

where, $k^2 = 1 - \frac{\beta^2 l^2}{\alpha^2}$. Inverting the above expression we obtain the form of $r(\sigma)$ which is given as :

$$r(\sigma) = l\ |sn\left[\pm\frac{\alpha(\sigma - \sigma_0)}{l}\right]| \tag{23}$$

where sn denotes the Jacobian elliptic function. This string solution lies exclusively inside the horizon of the black hole. However, as in the Minkowski spacetime solution, one needs to take the absolute value in order to satisfy the boundary conditions at the edges – this results in a similar *kink* in the metric and a δ function singularity at $\sigma = 0$. The embedding function is plotted in Fig. 1. The edges of the worldsheet are at those points where the derivative of the function $r(\sigma)$ vanishes.

The induced metric on the world-sheet (at points other than $\sigma = 0$) is given by,

$$ds_I^2 = -\alpha^2 dn^2(\bar{\sigma})[-d\tau^2 + d\sigma^2] \tag{24}$$

for the solution inside the horizon. The roles of τ and σ inside the horizon are reversed in a way similar to that of r and t for the background metric. The worldsheet is everywhere non–singular in this case except for the δ function in the Riemann curvature at $\sigma = 0$ $(r = 0)$.

One can construct a string solution which would reside entirely outside the horizon of the black hole. This turns out to be given as :

$$r(\sigma) = \frac{l}{k}\frac{dn(\bar{\sigma})}{cn(\bar{\sigma})} \tag{25}$$

with $\bar{\sigma} = \pm\frac{\alpha(\sigma - \sigma_0)}{l}$.

The embedding function is plotted in Fig. 2. It can be interpreted as a semi–infinite string with its only boundary at $\sigma = 0$ $(r = r_1)$. The boundary condition is obviously satisfied at

this edge which corresponds to a minimum of the embedding function (i.e. $r'(\sigma = 0) = 0$. Note the curious fact that the boundary is not exactly at the horizon (which correponds to $r = l = 1$ in Figure 2) but slightly above it (more precisely, at the value $r = \frac{l}{k}$).

The induced metric for this worldsheet is :

$$ds_I^2 = \beta^2 l^2 \frac{sn^2(\bar{\sigma})}{cn^2(\bar{\sigma})}[-d\tau^2 + d\sigma^2] \tag{26}$$

It is easy to see from the evaluation of the Ricci scalar that the worldsheet becomes singular at the only boundary of this semi–infinite string.

IV. PERTURBATIONS

A. Perturbations for planetoids in general spherically symmetric, static backgrounds

We now consider perturbative deformations of world-sheet in the manifestly covariant formalism [9]. The background metric is the usual, spherically symmetric, static one mentioned in section II. We have two unknown functions $b(r)$ and $\psi(r)$, which we specify according to our choice of the background.

The equations governing the perturbations are the Jacobi equations and are related to the second variation of the Nambu–Goto action evaluated at its stationary points. A general perturbation of the embedding function $x^\mu(\sigma, \tau)$ can be written as $\delta x^\mu = E^\mu_a \phi^a + n^\mu_i \phi^i$ where ϕ^a and $\phi^{(i)}$ are the perturbations along the a–th tangent and the i-th normal respectively. We ignore the tangential perturbations because they are essentially related to the reparametrisation of the worldsheet and do not cause any deformation of the worldsheet geometry, which is invariant under such transformations. The equations satisfied by the quantities ϕ^i are given as [9]:

$$\Box \phi^{(i)} + (M^2)^i_j \phi^{(j)} = 0 \tag{27}$$

However, it should be mentioned that in the most general setting, the second variation of the Nambu-Goto action does involve quantities like the components of the normal fundamental form $\mu^a_{ij} = g_{\mu\nu} n^\mu_i E^\rho_a D_\rho n^\nu_j$. With the choice of the embedding (planetoid) and the

normals given below (for $\delta = 0$) it can be shown quite easily that all components of the normal fundamental form are identically equal to zero irrespective of the specific form of the functions $b(r)$ and $\psi(r)$ in the background metric. Therefore we can use these equations for the $\phi^{(i)}$ in our analysis of the perturbations.

For strings in four dimensional backgrounds, these equations constitute a pair of coupled differential equations, with the quantity $(M^2)^{ij}$ given as :

$$\left(M^2\right)^{ij} = K^{abi}K^{j}_{ab} + R_{\alpha\beta\mu\nu}E^{\alpha a}n^{\beta i}E^{\mu}_{a}n^{\nu j} \tag{28}$$

with $K^{abi} = -g_{\mu\nu}\left(E^{\alpha}_{a}D_{\alpha}E^{\mu}_{b}\right)n^{\nu i}$, being the extrinsic curvature tensor of the worldsheet along the i–th normal direction, $R_{\mu\nu\rho\sigma}$ the Riemann tensor for the background spacetime, E^{μ}_{a} the tangents in an orthonormal frame and $n^{\mu i}$ the normals. D_a is the world-sheet projection of the space-time covariant derivative D_μ, where $D_a = E^{\mu}_{a}D_\mu$, μ, $\nu = 0, 1, \cdots N-1$; $a = 1, 2, \cdots D$, ($a = \tau$, σ for a string worldsheet); $i = 1, 2, \cdots N - D$ (where D is the number of world-sheet indices). We shall denote the second term in the R.H.S of the above expression in future discussion as A^{ij}. Evaluation of the quantity $(M^2)^{ij}$ thus depends on the general expressions for the tangents and normals to the worldsheet. These are taken to be,

$$E^{\mu}_{\tau} \equiv \left(\frac{\alpha}{\sqrt{e^{2\psi}\alpha^2 - \beta^2 r^2}}, 0, 0, \frac{\beta}{\sqrt{e^{2\psi}\alpha^2 - \beta^2 r^2}}\right) \quad ; \quad E^{\mu}_{\sigma} \equiv \left(0, \sqrt{1 - \frac{b}{r}}, 0, 0\right) \tag{29}$$

$$n^{\mu 1} \equiv \left(\frac{r\beta e^{-\psi}\sin\delta}{\sqrt{e^{2\psi}\alpha^2 - \beta^2 r^2}}, 0, \frac{1}{r}\cos\delta, \frac{\alpha e^{\psi}\sin\delta}{r\sqrt{e^{2\psi}\alpha^2 - \beta^2 r^2}}\right) \tag{30}$$

$$n^{\mu 2} \equiv \left(\frac{r\beta e^{-\psi}\cos\delta}{\sqrt{e^{2\psi}\alpha^2 - \beta^2 r^2}}, 0, -\frac{1}{r}\sin\delta, \frac{\alpha e^{\psi}\cos\delta}{r\sqrt{e^{2\psi}\alpha^2 - \beta^2 r^2}}\right) \tag{31}$$

where δ is an arbitrary angular parameter. For different values of δ we have different normals. However all of them are related to each other by $O(2)$ transformations. (In a general N dimensional background with a D dimensional object living in it there is an $O(N-D)$ gauge freedom in the choice of normals.) More specifically,

$$\bar{n}^{\mu i} = R^{ij}n^{\mu j} \tag{32}$$

where R^{ij} is the $O(N-D)$ dimensional rotation matrix (in our case we have an $O(2)$ matrix).

We shall work with $\delta = 0$ and then write down all expressions for a general δ by using its transformation properties.

The expressions for the K^i_{ab} are given as follows. The only nonzero components are the $K^2_{\tau\sigma}$ and $K^2_{\sigma\tau}$ which are ofcourse equal.

$$K^2_{\sigma\tau} = K^2_{\tau\sigma} = \frac{\alpha\beta e^{\psi}\sqrt{1-\frac{b}{r}}}{e^{2\psi}\alpha^2 - \beta^2 r^2}\left(r\psi' - 1\right) \tag{33}$$

We also need to evaluate the quantity $A^{ij} = R_{\alpha\beta\mu\nu}E^{\alpha a}n^{\beta i}E^{\mu}_{a}n^{\nu j}$

As can be seen easily, the A^{ij} for $i \neq j$ are all zero. For A^{11} and A^{22} we have the following expressions :

$$A^{11} = \frac{b'r - b}{2r^3} + \frac{1}{r\left(e^{2\psi}\alpha^2 - \beta^2 r^2\right)}\left[e^{2\psi}\psi'\left(1 - \frac{b}{r}\right)\alpha^2 - \beta^2 b\right] \tag{34}$$

$$A^{22} = -\frac{\psi'}{r}\left(1 - \frac{b}{r}\right) + \frac{e^{2\psi}\alpha^2}{e^{2\psi}\alpha^2 - \beta^2 r^2}\frac{b'r - b}{2r^3} + \frac{r^2\beta^2\left(1 - \frac{b}{r}\right)}{e^{2\psi}\alpha^2 - \beta^2 r^2}\left[\psi'' - \frac{b'r - b}{2r(r-b)}\psi' + \psi'^2\right] \tag{35}$$

We now need to write down the full expression for the quantity for $(M^2)^{ij}$ This turns out to be ,

$$\left(M^2\right)^{ij} = K^{abi}K^j_{ab} + A^{ij} \tag{36}$$

Since $(M^2)^{11}$ is equal to A^{11} we write down the expression for $(M^2)^{22}$ only.

$$\begin{aligned}\left(M^2\right)^{22} &= -2\left(\frac{\alpha\beta e^{\psi}\sqrt{1-\frac{b}{r}}}{e^{2\psi}\alpha^2 - \beta^2 r^2}\left(r\psi' - 1\right)\right)^2 - \frac{\psi'}{r}\left(1 - \frac{b}{r}\right) \\ &+ \frac{e^{2\psi}\alpha^2}{e^{2\psi}\alpha^2 - \beta^2 r^2}\frac{b'r - b}{2r^3} + \frac{r^2\beta^2\left(1 - \frac{b}{r}\right)}{e^{2\psi}\alpha^2 - \beta^2 r^2}\left[\psi'' - \frac{b'r - b}{2r(r-b)}\psi' + \psi'^2\right]\end{aligned} \tag{37}$$

Let us now move on towards writing down the expressions for a general δ. We shall denote quantities defined with respect to the new normal with an overbar. $K^i_{ab}K^{abj}$ is denoted as B^{ij}. Therefore we have :

$$\bar{B}^{11} = \sin^2\delta B^{22} \quad ; \quad \bar{B}^{12} = \sin\delta\cos\delta B^{22} \quad ; \quad \bar{B}^{22} = \cos^2\delta B^{22} \tag{38}$$

Similarly for $\bar{A}^{ij}$ we have :

$$\bar{A}^{11} = \cos^2\delta A^{11} + \sin^2\delta A^{22} \tag{39}$$

$$\bar{A}^{22} = \sin^2\delta A^{11} + \cos^2\delta A^{22} \tag{40}$$

$$\bar{A}^{12} = \sin\delta\cos\delta\left(A^{22} - A^{11}\right) \tag{41}$$

Now, by virtue of the presence of the off diagonal terms A^{12}, B^{12} we will have genuinely coupled equations which govern the perturbations of the planetoid solution. We shall however confine ourselves to $\delta = 0$ where the equations are uncoupled and easier to solve. It must be admitted however, that a completely general treatment of perturbations should be done with an arbitrary δ with δ depending on σ and τ as well. In the latter case, the above equations, which correspond to rigid rotations of the normal frame will naturally be modified.

(1) Minkowski spacetime in spherical coordinates

We now analyse the perturbations about the planetoid string configuration in spherically symmetric Minkowski spacetime. Recall that the string configuration (with $\sigma_0 = 0$) is given as :

$$t = t_0 + \alpha\tau \quad ; \quad r = \frac{\alpha}{\beta}|\sin\beta\sigma| \quad ; \quad \theta = \frac{\pi}{2} \quad ; \quad \phi = \phi_0 + \beta\tau \tag{42}$$

We shall confine ourselves to the domain of σ given as : $0 < \sigma < \frac{\pi}{2\beta}$ or $-\frac{\pi}{2\beta} < \sigma < 0$.

The tangents E^μ_a and normals $n^{\mu i}$ to the worldsheet are,

$$E^\mu_\tau \equiv \left(\frac{\alpha}{\sqrt{\alpha^2 - \beta^2 r^2}}, 0, 0, \frac{\beta}{\sqrt{\alpha^2 - \beta^2 r^2}}\right) \quad ; \quad E^\mu_\sigma \equiv (0, 1, 0, 0) \tag{43}$$

$$n^{\mu 1} \equiv \left(0, 0, \frac{1}{r}, 0\right) \quad ; \quad n^{\mu 2} \equiv \left(\frac{r\beta}{\sqrt{\alpha^2 - \beta^2 r^2}}, 0, 0, \frac{\alpha}{r\sqrt{\alpha^2 - \beta^2 r^2}}\right) \tag{44}$$

Note that $(E^\mu_a, n^{\mu i})$ form an orthonormal spacetime basis.

The nonzero K^i_{ab} are given as :

$$K^2_{\sigma\tau} = K^2_{\tau\sigma} = -\frac{\alpha\beta}{\alpha^2 - \beta^2 r^2} \tag{45}$$

Note that the extrinsic curvature tensor components also diverge at the edges of the string world–sheet. Therefore, if we evaluate the mean curvature we will find indeterminate quantities appearing at the edge values of σ. The Nambu–Goto string worldsheet is ill–defined at the edges.

Therefore we have,

$$\left(M^2\right)^{22} = K^{\sigma\tau 2}K^2_{\sigma\tau} + K^{\tau\sigma 2}K^2_{\tau\sigma} = -\frac{2\beta^2}{\alpha^2\cos^4\beta\sigma} \tag{46}$$

The other entitities in the $(M^2)^{ij}$ matrix are all zero. Hence the perturbation equation for $\phi^{(2)}$ is given as below :

$$-\frac{\partial^2\phi^{(2)}}{\partial\tau^2} + \frac{\partial^2\phi^{(2)}}{\partial\sigma^2} - \frac{2\beta^2}{\cos^2\beta\sigma}\phi^{(2)} = 0 \tag{47}$$

On separating variables we have the harmonic oscillator equation for the τ variable while the equation for the σ variable turns out to be :

$$\frac{d^2\Sigma}{d\sigma^2} + \left(\omega^2 - \frac{2\beta^2}{\cos^2\beta\sigma}\right)\Sigma = 0 \tag{48}$$

A simple pair of linearly independent solutions to the perturbation equation can be obtained for $\omega = 0$. This is given as :

$$\Sigma = \tan\beta\sigma \qquad ; \qquad \Sigma = \beta\sigma\tan\beta\sigma + 1 \tag{49}$$

Note that both these solutions have a divergence at $\sigma = \frac{\pi}{2\beta}$ This is due to the singular edges of the worldsheet geometry of the planetoid configuration. The lowest mode of perturbation results in a solution which blows up at the edges of the worldsheet. To obtain information about the higher modes one needs to look at the general solution of the relevant equation. General solutions for eigenvalues (ω_n^2) are known to exist for the Schrodinger equation with a potential $V(\sigma) = 2\beta^2 sec^2\beta\sigma$ in quantum mechanics. There, this potential is known as the Poschl–Teller I potential. The eigenvalues and eigenfunctions (unnormalised) are [17],

$$\omega_n^2 = 4n^2\beta^2 \tag{50}$$

$$\Sigma_n(\beta;\sigma) = \left(\frac{1-\gamma}{1+\gamma}\right)^{\frac{1}{2}} P_n^{\frac{1}{2},\frac{-3}{2}}(\gamma) \tag{51}$$

where the P_n denotes the Jacobi polynomial and $\gamma = 1 - 2\sin^2\beta\sigma$.

The explicit expressions for the higher values of n all contain an overall factor $\tan\beta\sigma$. Therefore, at $\sigma = \frac{\pi}{2\beta}$ there will always be a divergence which is caused by the singularity in the planetoid solution. If, however, as mentioned before we restrict ourselves to the domain $-\frac{\pi}{2\beta} < \sigma < 0$ or $0 < \sigma < \frac{\pi}{2\beta}$ where the Nambu–Goto string is well defined in both the extrinsic and intrinsic sense then, ofcourse, there is no problem with stability.

The perturbation $\phi^{(1)}$ satisfies a simple wave equation whose solutions are trivial. Note also the fact that the generalised Raychaudhuri equation for the planetoid strings would also be the same with the $\phi^{(2)}$ replaced by F ($\theta_a = \frac{\partial_a F}{F}$). Therefore, knowing the solutions of the Σ equation would imply solving both for perturbative as well as non–perturbative deformations of the string configuration.

(2) Spherical Rindler Spacetime

The perturbation equations for the planetoid string in spherical Rindler spacetime are given as (these are exclusively the equations for the σ part of the perturbation $\phi^{(i)} = T^{(i)}(\tau)\Sigma^{(i)}(\sigma)$. The τ part yields the usual harmonic oscillator equations. The σ equations are also harmonic oscillator equations with different constants acting as the spring constant. These are,

$$\frac{d^2\Sigma^{(1)}}{d\sigma^2} + \left(\omega^2 + \alpha^2\right)\Sigma^{(1)} = 0 \tag{52}$$

$$\frac{d^2\Sigma^{(2)}}{d\sigma^2} + \left(\omega^2 - \left(\alpha^2 - \beta^2\right)\right)\Sigma^{(2)} = 0 \tag{53}$$

The solutions to these equations are trivial. They are exponential or oscillatory according to the value of ω^2. For the $\Sigma^{(1)}$, we just need $\omega^2 > 0$ for oscillatory solutions. On the other hand, for $\Sigma^{(2)}$ we need $\omega^2 > (<)\alpha^2 - \beta^2$ for oscillatory (exponential) solutions.

V. SEMI-CLASSICAL QUANTIZATION

Semi-classical quantization of strings were performed in [18] following the prescription given by Dashen *etal* [19]for time periodic solutions in quantum mechanics and quantum field theory. In order to quantize the string solutions semi-classically, one needs to compute the classical action of solutions S_{cl} as a function of string mass m, where,

$$m = -\frac{dS_{cl}}{dT} \tag{54}$$

and T is the period in physical time given by, $T = \frac{2\pi\alpha}{\beta}$. Hence, a knowledge of classical solutions is necessary for semi-classical quantization. Using (3), the expression for the classical action of solutions is given as,

$$S_{cl} = -\frac{2T}{\pi\alpha'}\int_{r_{min}}^{r_{max}} dr g_{rr}\sqrt{-\tilde{V}(r)} \tag{55}$$

where, r_{min} and r_{max} are the minimum and maximum radius reached by the string respectively. The idea is to use the functional formulation of the WKB approximation via path integrals. The functional integral is evaluated in a stationary phase approximation, where one integrates over a function space and the stationary phase points are the functions which satisfy the classical equation of motion and are periodic solutions. The reduced action $W(m)$ is defined as,

$$W \equiv S_{cl}(T(m)) + mT(m) = \frac{4}{T\alpha'}\int_{r_{min}}^{r_{max}} dr \frac{Tg_{t\phi} + 2\pi g_{\phi\phi}}{\sqrt{-\tilde{V}(r)}} \tag{56}$$

The quantization condition is given by, $W = 2\pi n$. However, for the class of solutions being discussed here, the quantization condition is equivalent to

$$W = 2\pi J \tag{57}$$

where J is the string angular momentum obtained by integrating the conserved world-sheet current J_μ ($\mu = \tau, \sigma$). The expression for J is given by,

$$J = \frac{2}{\pi\alpha'}\int_{r_{min}}^{r_{max}} dr \frac{g_{t\phi} + \frac{2\pi}{T}g_{\phi\phi}}{\sqrt{-\tilde{V}(r)}} \tag{58}$$

We now compute these quantities for the Ellis geometry, spherical Rindler spacetime and the $2+1$ dimensional black hole backgrounds to obtain the spectrum. We write down the quantization conditions in each of these cases and the relation between mass and angular momentum turns out to lead to non-linear Regge trajectories ($\alpha' m^2 \neq 4J$). In Minkowski space, one gets a linear Regge trajectory [5]

(1) Ellis Geometry

For the case of Ellis geometry, the expressions for the physical quantities are given by,

$$S_{cl} = \frac{\pi}{\alpha'}\left(b_0^2 - \frac{\alpha^2}{\beta^2}\right) \tag{59}$$

$$W = \frac{\pi}{\alpha'}\left(b_0^2 + \frac{\alpha^2}{\beta^2}\right) \tag{60}$$

Therefore, the mass m is given by,

$$m = \frac{\alpha}{\alpha'\beta} \tag{61}$$

Evaluating $J = \frac{W}{2\pi}$ one gets the curious relation,

$$2J = \alpha' m^2 + \frac{b_0^2}{\alpha'} \tag{62}$$

This is a linear Regge relation with an intercept proportional to the square of the throat radius of the wormhole. It should be mentioned that by naively putting $b_0 = 0$ in the Ellis geometry one does not get back Minkowski spacetime – but a couple of Minkowski worlds connected by a throat which has gone singular by the assumption $b_0 = 0$. This is also reflected in the Regge relation where we notice a missing factor of 2 in comparison with the Minkowski spacetime result $4J = \alpha' m^2$.

(2) Spherical Rindler spacetime

The expressions S_{cl}, reduced action $W(m)$, mass and string angular momentum are given by,

$$S_{cl} = -\frac{T}{\pi\alpha'}\sqrt{1 - \frac{4\pi^2}{T^2}}; \qquad W(m) = \frac{4\pi}{T\alpha'}\frac{1}{\sqrt{1 - \frac{4\pi^2}{T^2}}} \tag{63}$$

And,

$$m = \frac{1}{\pi\alpha'\sqrt{1-\frac{4\pi^2}{T^2}}}; \qquad J = \frac{2}{T\alpha'\sqrt{1-\frac{4\pi^2}{T^2}}} \tag{64}$$

One can again note from the above relations that $\alpha' m^2$ is not proportional to J. Infact, as can be easily seen, $J = \frac{\beta}{\alpha} m$.

(3) 2 + 1 dimensional black hole space-time

The physical potential for $2+1$ dimensional black hole (assuming $M = 1, J = 0$) is given by,

$$\tilde{V}(r) = \left(1 - \frac{r^2}{l^2}\right)\left[\frac{r^2}{l^2} - 1 - \frac{4\pi^2 r^2}{T^2}\right] \tag{65}$$

In this case, there are two kinds of planetoid strings, one which starts from $r = 0$ and ends precisely at the horizon having a finite length. The other one which starts at a point away from the horizon and extends upto infinity with an infinite length. We shall consider the planetoid strings which are inside the horizon. For these strings, the maximum radius $r_{max} = l$ and $r_{min} = 0$. The classical action of the solution is,

$$S_{cl} = \frac{2Tl}{\pi\alpha'} E(k) \tag{66}$$

where $E(k)$ is the complete elliptic integral and $k = \sqrt{1 - \frac{4\pi^2 l^2}{T^2}}$ is the elliptic modulus [16]. W is given by the expression,

$$W = \frac{8\pi l^3}{T\alpha' k^2}\left[K(k) - E(k)\right] \tag{67}$$

We can now compute the mass m and angular momentum J from the above expressions and they are given by,

$$m = \frac{l}{\pi T^2 \alpha' k^2}\left[-2T^2 E(k) + 8\pi^2 l^2 K(k)\right] \tag{68}$$

$$J = \frac{4l^3}{T\alpha' k^2}\left[K(k) - E(k)\right] \tag{69}$$

Since J is not proportional to m^2 it leads to a nonlinear Regge trajectory. For $T \sim 2\pi l$, the quantization condition reads $T^2 \sim 4\pi^2 n\alpha'$. In the above limit, $k \to 0$ and the mass of the string becomes $\alpha' m^2 \simeq 4n$, where one recovers the linear behaviour of the Regge trajectory.

The invariant string length is given by,

$$s = 2\int_0^{r_{max}} dr\sqrt{g_{rr}} = \pi l \tag{70}$$

On the other hand, if we consider the other planetoid string where $r_{min} = \frac{l^2}{1-\frac{4\pi^2 l^2}{T^2}}$, then the string length becomes infinite as it stretches outside the horizon.

VI. SUMMARY AND CONCLUSIONS

Let us first summarise the results we have obtained. Firstly, we have new examples of planetoids in a variety of of spacetime backgrounds. These include the Ellis geometry, the spherical Rindler geometry and the spacetime of the $2+1$ dimensional BTZ black hole. A curious feature about most of these planetoid string configurations is the presence of a *world-sheet singularity* at the edges. Thereafter, we write down the general equations governing perturbations of any planetoid solution in a static, spherically background. The general formalism is then applied to planetoids in Minkowski and spherical Rindler spacetimes, where the planetoid perturbation equations turn out to be exactly solvable. We comment on the stability of these solutions by looking at divergences in the solutions governing perturbations. The Minkowski spacetime planetoid solution turns out to be stable in the nonsingular domain of the worldsheet geometry where we can apply the perturbation theory. The planetoid in spherical Rindler spacetime is also stable. We have not been able to integrate the perturbation equation for the planetoids in a $2+1$ dimensional black hole or the Ellis geometry essentially due to the complicated form of the string configurations. Finally, using methods of semiclassical quantisation we proceed to quantise the planetoids obtained in Section III. The Regge relations are written down and the semiclassical quantisation conditions are derived for each of these planetoids. It turns out that in the Ellis geometry, the Regge

relation is linear but we have an intercept proportional to the square of the wormhole throat radius. In spherical Rindler spacetime J is linearly related to m while in the BTZ black hole we cannot write down an explicit general relation between J and m.

Obtaining further planetoid solutions would be obviously of interest in future. A generalisation of the ansatz which could be applicable in time-dependent backgrounds would also be worth attempting (recall that in Section I, we showed that the planetoid ansatz is incompatible with time–dependant metrics of a certain general type). The recent work on rigidly rotating strings [6] proposes a more general class of strings of which planetoids are a special case. It would be worth trying out new examples of such rigidly rotating strings in curved backgrounds and analyse the stability of these configurations. Finally, the question of non–perturbative deformations which are governed by the generalised Raychaudhuri equations must be addressed in order to study the formation of cusps and kinks on the string worldsheet. Planetoids and rigidly rotating strings could be prototypes where these equations are solvable and information can be obtained in atleast a specific context.

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FIGURE CAPTIONS

Fig. 1 : The embedding function $r(\sigma)$ versus σ for the planetoid string inside the horizon of the BTZ black hole. The values for the various parameters are : $\alpha = 1$, $\beta = .5$, $l = 1$, $k^2 = .75$

Fig. 2 : The embedding function $r(\sigma)$ versus σ for the planetoid string outside the horizon of the BTZ black hole. The values for the various parameters are : $\alpha = 1$, $\beta = .5$, $l = 1$, $k^2 = .75$

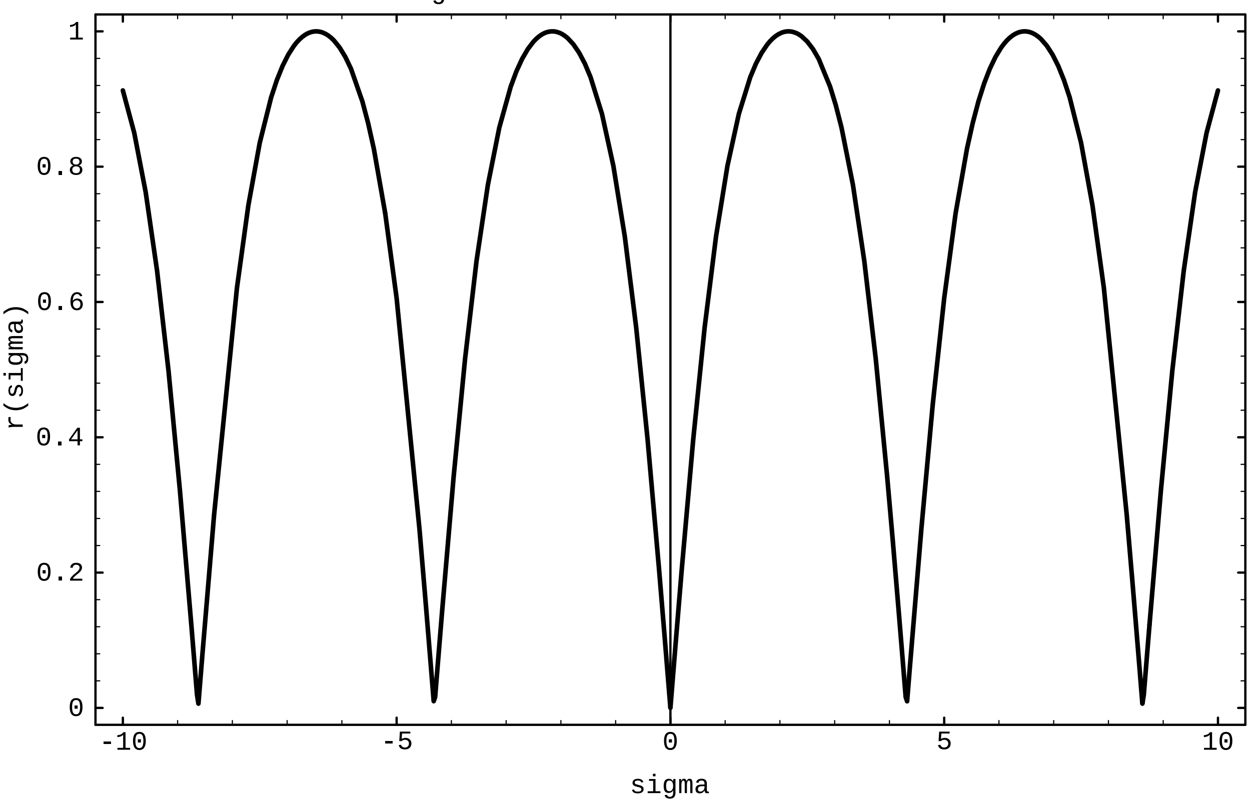

Figure 1 : Solution inside horizon

Figure 2 : Solution outside horizon